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Pseudo-Riemannian Novikov algebras

Zhiqi Chen and Fuhai Zhu

School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, People's Republic of China

E-mail: chenzhiqi@nankai.edu.cn and zhufuhai@nankai.edu.cn

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Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type and Hamiltonian operators in formal variational calculus. Pseudo-Riemannian Novikov algebras denote Novikov algebras with non-degenerate invariant symmetric bilinear forms. In this paper, we find that there is a remarkable geometry on pseudo-Riemannian Novikov algebras, and give a special class of pseudo-Riemannian Novikov algebras.

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1. Introduction

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type [1–3] and Hamiltonian operators in formal variational calculus [4–9]. There has been much progress in the study of Novikov algebras [10–20]. A Novikov algebra A is a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \mapsto xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{1}$$

$$(xy)z = (xz)y \tag{2}$$

for any $x, y, z \in A$. Novikov algebras are a special class of left symmetric algebras. Left symmetric algebras are non-associative algebras obeying equation (1). They arise from the study of affine manifolds, affine structures and convex homogeneous cones [21–25]. A fermionic Novikov algebra was introduced as a left-symmetric algebra with anti-commutative right multiplication operators: an algebra is a fermionic Novikov algebra if its product satisfies equation (1) and

$$(xy)z = -(xz)y. \tag{3}$$

The commutator of a left-symmetric algebra A, i.e.,

$$[x, y] = xy - yx, \tag{4}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the underlying Lie algebra of A. The underlying Lie algebra is said to admit a left-symmetric structure, or an affine structure. An affine structure on a Lie algebra $\mathfrak{g}$ with Lie product $[\cdot,\cdot]$ is a bilinear product $(x, y) \mapsto xy$ satisfying equations (1) and (4). Furthermore, if the product satisfies equation (2) (or equation (3)), we say that $\mathfrak{g}$ admits a Novikov (or fermionic Novikov) structure. Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra, i.e.,

$$[x, [y, z]] = 0, \qquad \forall\, x, y, z \in \mathfrak{g}.$$

Then the bilinear product on $\mathfrak{g}$ defined by

$$xy = \tfrac{1}{2}[x, y]$$

is both a Novikov structure and a fermionic Novikov structure on $\mathfrak{g}$. Also there are other Novikov and fermionic Novikov structures. In fact, let $\mathfrak{g}$ be a Lie algebra of dimension 4 with a basis e_1, e_2, e_3, e_4 satisfying

$$[e_2, e_4] = e_1, \qquad [e_4, e_3] = e_2.$$

It is easy to see that the bilinear product on $\mathfrak{g}$ defined by

$$e_2e_4 = -e_3e_3 = e_1, \qquad e_4e_3 = e_2, \qquad e_4e_4 = e_3$$

is not a Novikov structure but a fermionic Novikov structure on $\mathfrak{g}$ and the bilinear product on $\mathfrak{g}$ defined by

$$e_2e_4 = -e_3e_3 = e_1, \qquad e_3e_4 = 2e_4e_3 = -2e_2, \qquad e_4e_4 = e_3$$

is not a fermionic Novikov structure but a Novikov structure on $\mathfrak{g}$.

A pseudo-Riemannian connection is a pseudo-metric connection such that the torsion is zero and parallel translation preserves the bilinear form on the tangent spaces [26]. The corresponding structure on the algebra A is a non-degenerate symmetric bilinear form $f : A \times A \to \mathbb{F}$ satisfying

$$f(xy, z) + f(y, xz) = 0 \tag{5}$$

for any $x, y, z \in A$. If the algebra A is a (fermionic) Novikov algebra, then (A, f) is called a pseudo-Riemannian (fermionic) Novikov algebra.

In this paper, we show that the underlying Lie algebra of a pseudo-Riemannian Novikov algebra is a pseudo-Riemannian Lie algebra, which was first introduced in [27] and strongly related to pseudo-Riemannian Poisson manifolds [28] with a compatible pseudo-metric. In view of lack of examples, we give a class of pseudo-Riemannian Novikov algebras.

The paper is organized as follows. In section 2, we show that the underlying Lie algebra of a pseudo-Riemannian Novikov algebra is the Lie algebra obtained by linearizing the Poisson structure at a point of a Poisson manifold with a compatible pseudo-metric and a certain condition on the Levi-Civita contravariant connection. In section 3, we give the classification of pseudo-Riemannian Novikov algebras (A, f) with $\dim AA = \dim A - 1$ over the complex field $\mathbb{C}$. Based on the discussion in the previous sections, we get some conclusions in section 4.

2. Pseudo-Riemannian Novikov algebras

In this section, the algebras are of finite dimension and over $\mathbb{R}$. A Lie algebra $\mathfrak{g}$ with Lie product $[\cdot,\cdot]$ is called a pseudo-Riemannian Lie algebra if there is a bilinear product $(x, y) \mapsto xy$ on $\mathfrak{g}$ satisfying

$$[x, y] = xy - yx, \qquad [xz, y] + [x, yz] = 0$$

and a non-degenerate symmetric bilinear form f satisfying

$$f(xy, z) + f(y, xz) = 0$$

for any $x, y, z \in \mathfrak{g}$. Furthermore, if f is positive definite, $\mathfrak{g}$ is called a Riemann–Lie algebra. The following are some important applications of pseudo-Riemannian Lie algebras [28–30].

(i) If $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra, then there is a pseudo-metric $\langle , \rangle$ on the dual $\mathfrak{g}^*$ endowed with its linear Poisson structure π for which the triple $(\mathfrak{g}^*, \pi, \langle , \rangle)$ is a pseudo-Riemannian Poisson manifold.

(ii) If $(P, \pi, \langle , \rangle)$ is a Riemann–Poisson manifold and S is a symplectic leaf of P, then S is a Kähler manifold.

(iii) If $\mathfrak{g}$ is a Riemann–Lie algebra, then any even dimensional subalgebra of the orthogonal subalgebra defined in [29] gives rise to a structure of a Riemann–Poisson Lie group on any Lie group whose Lie algebra is $\mathfrak{g}$. Moreover, we get a structure of Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ where both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Riemann–Lie algebras.

(iv) The underlying Lie algebra of a pseudo-Riemannian fermionic Novikov algebra is a pseudo-Riemannian Lie algebra.

The notion of pseudo-Riemannian Lie algebras was first introduced in [27]. They are strongly related to pseudo-Riemannian Poisson manifolds [28]. In fact, let P be a Poisson manifold with a Poisson tensor π. A pseudo-metric of signature (p, q) on the cotangent bundle T^*P is a smooth symmetric contravariant 2-form $\langle , \rangle$ on P such that, at each point $o \in P$, $\langle , \rangle_o$ is non-degenerate on T_o^*P with signature (p, q). For any pseudo-metric $\langle , \rangle$ on T^*P, define a contravariant connection by

$$\begin{aligned}2\langle D_\alpha\beta, \gamma\rangle &= \sigma_\pi(\alpha).\langle\beta, \gamma\rangle + \sigma_\pi(\beta).\langle\alpha, \gamma\rangle - \sigma_\pi(\gamma).\langle\alpha, \beta\rangle \\ &\quad + \langle[\alpha, \beta]_\pi, \gamma\rangle + \langle[\gamma, \alpha]_\pi, \beta\rangle + \langle[\gamma, \beta]_\pi, \alpha\rangle\end{aligned}$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$ and the bundle map $\sigma_\pi : T^*P \to TP$ is defined by

$$\beta(\sigma_\pi(\alpha)) = \pi(\alpha, \beta), \qquad \forall\, \alpha, \beta \in T^*P,$$

and where the Lie bracket $[\cdot, \cdot]$ is given by

$$\begin{aligned}[\alpha, \beta]_\pi &= L_{\sigma_\pi(\alpha)}\beta - L_{\sigma_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) \\ &= i_{\sigma_\pi(\alpha)}d\beta - i_{\sigma_\pi(\beta)}d\alpha + d(\pi(\alpha, \beta)).\end{aligned}$$

Furthermore, if

$$\pi(D_\alpha df, \beta) + \pi(\alpha, D_\beta df) = 0 \tag{6}$$

for any $\alpha, \beta \in \Omega^1(P)$ and $f \in C^\infty(P)$, then the triple $(P, \pi, \langle , \rangle)$ is called a pseudo-Riemannian Poisson manifold. When $\langle , \rangle$ is positive definite we call the triple a Riemann–Poisson manifold. Let f denote the restriction of $\langle , \rangle$ on $\operatorname{Ker}\pi(o)$. Then, for any point $o \in P$ such that f is non-degenerate, the Lie algebra $\mathfrak{g}_o$ obtained by linearizing the Poisson structure at o is a pseudo-Riemannian Lie algebra.

Theorem 1. *Let notations be as above. If the Levi-Civita contravariant connection D mentioned above satisfies*

$$D_{D_\alpha\beta}\gamma = D_{D_\alpha\gamma}\beta \tag{7}$$

for any $\alpha, \beta, \gamma \in \Omega^1(P)$, then $\mathfrak{g}_o$ admits a Novikov structure.

Proof. For any $x, y \in \mathfrak{g}_o$ and $\alpha, \beta \in \Omega^1(P)$ such that $\alpha_o = x$ and $\beta_o = y$, it is easy to check that

$$(D_\alpha\beta)_o = xy.$$

It follows that

$$\left(D_{D_\alpha\beta}\gamma\right)_o = (D_\alpha\beta)_o z = (xy)z$$

for any $x, y, z \in \mathfrak{g}_o$ and $\alpha, \beta, \gamma \in \Omega^1(P)$ such that $\alpha_o = x$ and $\beta_o = y$ and $\gamma_o = z$. Thus, by equation (7), we have

$$(xy)z = (xz)y, \qquad \forall\, x, y, z \in \mathfrak{g}_o. \tag{8}$$

It follows that

$$\begin{aligned} 2f((xy)z, d) &= f((xy)z, d) + f((xy)z, d) \\ &= -f(z, (xy)d) + f(d, (xy)z) \\ &= -f(z, (xd)y) + f(d, (xz)y) \\ &= f((xd)z, y) - f((xz)d, y) \\ &= f((xd)z - (xd)z, y) \\ &= 0 \end{aligned}$$

for any $x, y, z, d \in \mathfrak{g}_o$. Therefore

$$(xy)z = 0$$

by the non-degeneracy of f. By the definition of pseudo-Riemannian Lie algebra,

$$(xy)z - x(yz) = -x(yz) = -y(xz) = (yx)z - y(xz). \tag{9}$$

Namely $\mathfrak{g}_o$ admits a Novikov structure. □

Corollary 1. *The underlying Lie algebras of pseudo-Riemannian Novikov algebras are pseudo-Riemannian Lie algebras.*

Proof. Let (A, f) be a pseudo-Riemannian Novikov algebra. By the proof of theorem 1, we have that

$$(xy)z = 0$$

for any $x, y, z \in A$. It follows that

$$\begin{aligned} [xz, y] + [x, yz] &= (xz)y - y(xz) + x(yz) - (yz)x \\ &= x(yz) - y(xz) \\ &= x(yz) - (xy)z - y(xz) + (yx)z \\ &= 0. \end{aligned}$$

Therefore, the underlying Lie algebra of A is a pseudo-Riemannian Lie algebra. □

Corollary 2. *Pseudo-Riemannian Novikov algebras are fermionic Novikov algebras.*

Therefore, the procedure to classify pseudo-Riemannian fermionic Novikov algebras in [30] is suitable to classify pseudo-Riemannian Novikov algebras. Moreover, by the classification of pseudo-Riemannian fermionic Novikov algebras up to dimension 4 in [30], we have

Corollary 3. *Pseudo-Riemannian fermionic Novikov algebras up to dimension 4 are Novikov algebras.*

Remark 1. But for dimensions greater that four, we could neither prove that pseudo-Riemannian fermionic Novikov algebras are Novikov algebras nor find a pseudo-Riemannian fermionic Novikov algebra which is not a Novikov algebra.

3. Pseudo-Riemannian Novikov algebras (A, f) with dim AA = dim $A - 1$

It is well known that the classification of Novikov algebras in higher dimensions is also an open problem. In fact, the classification in dimension 4 is not completely given. And the way to classify pseudo-Riemannian fermionic Novikov algebras [30] is only a theoretical method. In this section, we give a class of pseudo-Riemannian Novikov algebras over the field of complex numbers.

Theorem 2. *Let A be a Novikov algebra of dimension n with* $\dim AA = n-1$ *and f a symmetric bilinear form on A. Then (A, f) is a pseudo-Riemannian Novikov algebra if and only if n is odd and there exists an $(n-1)\times(n-1)$ anti-symmetric matrix M with* $\det M \neq 0$ *and a basis $\{e_1, e_2, \ldots, e_n\}$ of A such that either the product is given by*

$$(e_ne_1, \ldots, e_ne_{n-1}) = (e_1, \ldots, e_{n-1})M$$

and the bilinear form f is given by $F = I_n$ or the product is given by

$$(e_ne_1, \ldots, e_ne_n) = (e_1, \ldots, e_n)M_1$$

and the bilinear form f is given by

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 0 & \mathrm{row}_{n-1}(M) \\ 0 & \mathrm{row}_1(M) \\ \vdots & \vdots \\ 0 & \mathrm{row}_{n-2}(M) \\ 0 & 0 \end{pmatrix}.$$

Here $F = (f(e_i, e_j))$.

Firstly, we list some notations and establish some lemmas.

Let (A, f) be a pseudo-Riemannian Novikov algebra with a basis $\{e_1, e_2, \ldots, e_n\}$ and $\{c_{ij}^k\}$ the set of structure constants under the basis, i.e.,

$$e_ie_j = \sum_k c_{ij}^k e_k.$$

Denote the (form) character matrix by

$$\begin{pmatrix} \sum_k c_{11}^k e_k & \cdots & \sum_k c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_k c_{n1}^k e_k & \cdots & \sum_k c_{nn}^k e_k \end{pmatrix}.$$

Let AA denote the algebra defined by the linear span of the elements of the form xy for any $x, y \in A$ and define

$$LZ(A) = \{x \in A \mid xy = 0, \forall\, y \in A\},$$
$$RZ(A) = \{x \in A \mid yx = 0, \forall\, y \in A\},$$
$$(AA)^{\perp} = \{x \in A \mid f(x, yz) = 0, \forall\, y, z \in A\}.$$

Lemma 1. *Let* (A, f) *be a pseudo-Riemannian Novikov algebra with* $\dim A = \dim AA + 1$. *Then* $AA \subset LZ(A)$ *and* $\dim RZ(A) = 1$.

Proof. Since (A, f) is a pseudo-Riemannian Novikov algebra, then $(xy)z = 0, \forall\, x, y, z \in A$. Namely, $AA \subset LZ(A)$.

And it is easy to check that $RZ(A) = (AA)^{\perp}$, which implies $\dim A = \dim AA + \dim RZ(A)$. It follows that $\dim RZ(A) = 1$. □

As a sequence, we have $RZ(A) \cap AA = 0$ or $RZ(A) \subset AA$.

Lemma 2. *Let* (A, f) *be a pseudo-Riemannian Novikov algebra of dimension n with* $\dim AA = n - 1$. *If* $RZ(A) \cap AA = 0$*, then n is odd and there exist an* $(n-1) \times (n-1)$ *anti-symmetric matrix M with* $\det M \neq 0$ *and a basis* $\{e_1, e_2, \ldots, e_n\}$ *of A such that the product is given by*

$$(e_n e_1, \ldots, e_n e_{n-1}) = (e_1, \ldots, e_{n-1})M$$

and the bilinear form f is given by $F = I_n$.

Proof. Since $RZ(A) \cap AA = 0$, then $A = AA + RZ(A)$ and both $f|_{RZ(A)\times RZ(A)}$ and $f|_{AA\times AA}$ are non-degenerate. Thus, there exists a basis $\{e_1, \ldots, e_n\}$ of A such that the character matrix is given by

$$\begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ \sum_{k=1}^{n-1} c_{n1}^k e_k & \cdots & \sum_{k=1}^{n-1} c_{n(n-1)}^k e_k & 0 \end{pmatrix}$$

and f is defined by

$$F = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix},$$

where e_n is a basis of $RZ(A)$ and $\{e_1, \ldots, e_{n-1}\}$ is a basis of AA. It follows that the product is given by

$$(e_n e_1, \ldots, e_n e_{n-1}) = (e_1, \ldots, e_{n-1})M, \tag{10}$$

where

$$M = \begin{pmatrix} c_{n1}^1 & \cdots & c_{n(n-1)}^1 \\ \vdots & \ddots & \vdots \\ c_{n1}^{n-1} & \cdots & c_{n(n-1)}^{n-1} \end{pmatrix}.$$

Since $f(e_n e_i, e_j) + f(e_i, e_n e_j) = 0$, then

$$c_{ni}^j + c_{nj}^i = 0.$$

It follows that M is an anti-symmetric matrix. Since $\dim AA = n - 1$, then

$$\det M \neq 0.$$

But $\det M = 0$ if n is even. Thus n is odd. □

Lemma 3. *Let* (A, f) *be a pseudo-Riemannian Novikov algebra of dimension n with* $\dim AA = n - 1$. *If* $RZ(A) \subset AA$*, then n is odd and there exist an* $(n-1) \times (n-1)$

anti-symmetric matrix M with $\det M \neq 0$ *and a basis* $\{e_1, e_2, \ldots, e_n\}$ *of A such that the product is given by*

$$(e_n e_1, \ldots, e_n e_n) = (e_1, \ldots, e_n)M_1$$

and the bilinear form f is given by

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 0 & \mathrm{row}_{n-1}(M) \\ 0 & \mathrm{row}_1(M) \\ \vdots & \vdots \\ 0 & \mathrm{row}_{n-2}(M) \\ 0 & 0 \end{pmatrix}.$$

Proof. Since $RZ(A) \subset AA$, then $f|_{RZ(A)\times RZ(A)} = 0$. Furthermore, there exists a basis $\{e_1, \ldots, e_n\}$ of A such that the character matrix is given by

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & \sum_{k=1}^{n-1} c_{n2}^k e_k & \cdots & \sum_{k=1}^{n-1} c_{nn}^k e_k \end{pmatrix}$$

and f is defined by

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where e_1 is a basis of $RZ(A)$ and $\{e_1, \ldots, e_{n-1}\}$ is a basis of AA. It follows that the product is given by

$$(e_n e_1, \ldots, e_n e_n) = (e_1, \ldots, e_n)M_1, \tag{11}$$

where

$$M_1 = \begin{pmatrix} 0 & c_{n2}^1 & \cdots & c_{nn}^1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{n2}^{n-1} & \cdots & c_{nn}^{n-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $f(e_n e_i, e_j) + f(e_i, e_n e_j) = 0$, then

$$c_{ni}^j + c_{nj}^i = 0, \qquad i = 2, \ldots, n, \qquad j = 2, \ldots, n-1;$$
$$c_{ni}^1 + c_{nn}^i = 0, \qquad i = 2, \ldots, n.$$

It follows that

$$M = \begin{pmatrix} c_{n2}^2 & \cdots & c_{nn}^2 \\ \vdots & \ddots & \vdots \\ c_{n2}^{n-1} & \cdots & c_{nn}^{n-1} \\ c_{n2}^1 & \cdots & c_{nn}^1 \end{pmatrix}$$

is anti-symmetric. Since $\dim AA = n - 1$, then

$$\det M \neq 0.$$

But $\det M = 0$ if n is even. Therefore, n is odd. □

The proof of theorem 2:

$\Rightarrow$ The conclusion is clear thanks to Lemma 2 and 3.

$\Leftarrow$ For any case, $(e_i e_j)e_k = 0$. If $i \neq n$ and $j \neq n$, $e_i(e_j e_k) = 0$. Thus, A is a Novikov algebra. Since $\det M \neq 0$, then $\dim AA = n-1 = \dim A-1$. Obviously, f is non-degenerate. And it is easy to check that f satisfies equation (5). □

4. Conclusions

According to the discussion in the previous sections, we obtain some conclusions on pseudo-Riemannian Novikov algebras.

(i) The underlying Lie algebra of a pseudo-Riemannian Novikov algebra is the Lie algebra obtained by linearizing the Poisson structure at a point of a Poisson manifold with a compatible pseudo-metric and a certain condition (7) on the Levi-Civita contravariant connection.

(ii) Not every Novikov algebra is a fermionic Novikov algebra, but pseudo-Riemannian Novikov algebras are pseudo-Riemannian fermionic Novikov algebras. Moreover, pseudo-Riemannian Novikov algebras are equivalent with pseudo-Riemannian fermionic Novikov algebras in less than or equal to four dimensions.

(iii) If (A, f) is a pseudo-Riemannian Novikov algebra with $\dim AA = \dim A - 1$ over $\mathbb{C}$, then $\dim A = 2k + 1$. In fact, the structure of A is determined by an anti-symmetric matrix.

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